

**Robert E. Lucas, Jr. "Expectations and the Neutrality of Money,"**  
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## **I Introduction**

This paper provides a simple example of an economy in which equilibrium prices and quantities exhibit what may be the central feature of the modern business cycle: a systematic relation between the rate of change in nominal prices and the level of real output. The relationship, essentially a variant of the well-known Phillips curve, is derived within a framework from which all forms of "money illusion" are rigorously excluded: all prices are market-clearing, all agents behave optimally in light of their objectives and expectations, and expectations are formed optimally (in a sense to be made precise below).

Exchange in the economy studied takes place in two physically separated markets. The allocation of traders across markets in each period is in part stochastic, introducing fluctuations in relative prices between the two markets. A second source of disturbance arises from stochastic changes in the quantity of money, which in itself introduces fluctuations in the nominal price level (the average rate of exchange between money and goods). Information on the current state of these real and monetary disturbances is transmitted to agents only through prices in the market where each agent happens to be. In the particular framework presented below, prices convey this information only imperfectly, forcing agents to hedge on whether a particular price movement results from a relative demand shift or a nominal (monetary) one. This hedging behavior results in a nonneutrality of money, or broadly speaking a Phillips curve, similar in nature to that which we observe in reality. At the same time, classical results on the long-run neutrality of money, or independence of real and nominal magnitudes, continue to hold.

These features of aggregate economic behavior, derived below within a particular, abstract framework, bear more than a surface resemblance to many of the characteristics attributed to the U.S. economy by Friedman [3 and elsewhere]. This paper provides an explicitly elaborated example, to my knowledge the first, of an economy in which some of these propositions can be formulated rigorously and shown to be valid.

A second, in many respects closer, forerunner of the approach taken here is provided by Phelps. Phelps [8] foresees a new inflation and employment theory in which Phillips curves are obtained within a framework which is neoclassical except for "the removal of the postulate that all transactions are made under complete information." This is precisely what is attempted here.

The substantive results developed below are based on a concept of equilibrium which is, I believe, new (although closely related to the principles underlying dynamic programming) and which may be of independent interest. In this paper, equilibrium prices and quantities will be characterized mathematically as functions defined on the space of possible states of the economy, which are in turn characterized as finite dimensional vectors. This characterization permits a treatment of the relation of information to expectations which is in some ways much more satisfactory than is possible with conventional adaptive expectations hypotheses.

The physical structure of the model economy to be studied is set out in the following section. Section 3 deals with preference and demand functions; and in section 4, an exact definition of equilibrium is provided and motivated. The characteristics of this equilibrium are obtained in section 5, with certain existence and uniqueness arguments deferred to the appendix. The paper concludes

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\* I would like to thank James Scott for his helpful comments.

with the discussion of some of the implications of the theory in sections 6, 7, and 8.

## 2 The Structure of the Economy

In order to exhibit the phenomena described in the introduction, we shall utilize an abstract model economy, due in many of its essentials to Samuelson [10].<sup>1</sup> Each period,  $N$  identical individuals are born, each of whom lives for two periods (the current one and the next). In each period, then, there is a constant population of  $2N$ :  $N$  of age 0 and  $N$  of age I. During the first period of life, each person supplies, at his discretion,  $n$  units of labor which yield the same  $n$  units of output. Denote the output consumed by a member of the younger generation (its producer) by  $c^0$ , and that consumed by the old by  $c^1$ . Output cannot be stored but can be freely disposed of, so that the aggregate production-consumption possibilities for any period are completely described (in per capita terms) by:

$$c^0 + c^1 \leq n, \quad c^0, c^1, n \geq 0. \quad (1)$$

Since  $n$  may vary, it is physically possible for this economy to experience fluctuations in real output.

In addition to labor-output-, there is one other good: fiat money, issued by a government which has no other function. This money enters the economy by means of a beginning-of-period transfer to the members of the older generation, in a quantity proportional to the pretransfer holdings of each. No inheritance is possible, so that unspent cash balances revert, at the death of the holder, to the monetary authority.

Within this framework, the only exchange which can occur will involve a surrender of output by the young, in exchange for money held over from the preceding period, and altered by transfer, by the old.<sup>2</sup> We shall assume that such exchange occurs in two physically separate markets. To keep matters as simple as possible, we assume that the older generation is allocated across these two markets so as to equate total monetary demand between them. The young are allocated stochastically, fraction  $\theta/2$  going to one and  $1 - (\theta/2)$  to the other. Once the assignment of persons to markets is made, no switching or communication between markets is possible. Within each market, trading by auction occurs, with all trades transacted at a single, market clearing price.<sup>3</sup>

The pretransfer money supply, per member of the older generation, is known to all agents.<sup>4</sup> Denote this quantity by  $m$ . Posttransfer balances, denoted by  $m'$ , are not generally known (until next period) except to the extent that they are "revealed" to traders by the current period price level. Similarly, the allocation variable  $\theta$  is unknown, except indirectly via price. The development through time of the nominal money supply is governed by

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<sup>1</sup> The usefulness of this model as a framework for considering problems in monetary theory is indicated by the work of Cass and Yaari [1, 2].

<sup>2</sup> This is not quite right. If members of the younger generation were risk preferrers, they could and would exchange claims on future consumption among themselves so as to increase variance. This possibility will be filed out in the next section.

<sup>3</sup> This device of viewing traders as randomly allocated over distinct markets serves two purposes. First, it provides a setting in which information is imperfect in a specific (and hence analyzable) way. Second, random in the allocation of traders provides a source of relative price variation. This could as well have been achieved by postulating random taste or technology shifts, with little effect on the structure of the model.

<sup>4</sup> This somewhat artificial assumption, like the absence of capital goods and the serial independence of shocks, is part of an effort to keep the laws governing the transition of the economy from state to state as simple as possible. In general, I have tried to abstract from all sources of persistence of fluctuations, in order to focus on the nature of the initial disturbances.

$$m' = m x, \tag{2}$$

where  $x$  is a random variable. Let  $x'$  denote next period's value of this transfer variable, and let  $\theta'$  be next period's allocation variable. It is assumed that  $x$  and  $x'$  are independent, with the common, continuous density function  $f$  on  $(0, \infty)$ . Similarly,  $\theta$  and  $\theta'$  are independent, with the common, continuous symmetric density  $g$  on  $(0, 2)$ .

To summarize, the state of the economy in any period is entirely described by three variables  $m$ ,  $x$ , and  $\theta$ . The motion of the economy from state to state is independent of decisions made by individuals in the economy, and is given by (2) and the densities  $f$  and  $g$  of  $x$  and  $\theta$ .

### 3 Preferences and Demand Functions

We shall assume that the members of the older generation prefer more consumption to less, other things equal, and attach no utility to the holding of money. As a result, they will supply their cash holdings, as augmented by transfers, inelastically. (Equivalently, they have a unit elastic demand for goods.) The young, in contrast, have a nontrivial decision problem, to which we now turn.

The objects of choice for a person of age 0 are his current consumption  $c$ , current labor supplied,  $n$ , and future consumption, denoted by  $c'$ . All individuals evaluate these goods according to the common utility function:

$$U(c, n) + E\{V(c')\}. \tag{3}$$

(The distribution with respect to which the expectation in (3) is taken will be specified later.) The function  $U$  is increasing in  $c$ , decreasing in  $n$ , strictly concave, and continuously twice differentiable. In addition, current consumption and leisure are not inferior goods, or:

$$U_{cn} + U_{nn} < 0 \text{ and } U_{cc} + U_{cn} < 0. \tag{4}$$

The function  $V$  is increasing, strictly concave and continuously twice differentiable. The function  $V'(c')c'$  is increasing, with an elasticity bounded away from infinity, or:

$$V''(c')c' + V'(c') > 0, \tag{5}$$

$$c'V''(c')/V'(c') \leq -a < 0. \tag{6}$$

Condition (5) essentially insures that a rise in the price of future goods will, ceteris paribus, induce an increase in current consumption or that the substitution effect of such a price change will dominate its income effect.<sup>5</sup> The strict concavity requirement imposed on  $V$  implies that the left term of (6) be negative, so that (6) is a slight strengthening of concavity. Finally, we require that the marginal utility of future consumption be high enough to justify at least the first unit of labor expended, and ultimately tend to zero:

$$\lim_{c' \rightarrow 0} V'(c') = +\infty \tag{7}$$

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<sup>5</sup> The restrictions (4) and (5) are similar to those utilized in an econometric study of the labor market conducted by Rapping and myself, [5]. Their function here is the same as it was in [5]: to assure that the Phillips curve slopes the "right way."

$$\lim_{c' \rightarrow \infty} V'(c') = 0. \quad (8)$$

Future consumption,  $c'$ , cannot be purchased directly by an age 0 individual. Instead, a known quantity of nominal balances  $X$  is acquired in exchange for goods. If next period's price level (dollars per unit of output) is  $p'$  and if next period's transfer is  $x'$ , these balances will then purchase  $x'\lambda/p'$  units of future consumption.<sup>6</sup> Although it is purely formal at this point, it is convenient to have some notation for the distribution function of  $(x', p')$ , conditioned on the information currently available to the age-0 person: denote it by  $F(x', p' | m, p)$ , where  $p$  is the current price level. Then the decision problem facing an age-0 person is:

$$\max \{U(c, n) + S \int V'(x' \lambda / p') dF(x', p' | m, p)\} \quad (9)$$

subject to:

$$p(n - c) - \lambda \geq 0. \quad (10)$$

Provided the distribution  $F$  is so specified that the objective function is continuously differentiable, the Kuhn-Tucker conditions apply to this problem and are both necessary and sufficient. These are:

$$U_c(c, n) - p\mu \leq 0, \text{ with equality if } c > 0, \quad (11)$$

$$U_n(c, n) + p\mu \leq 0, \text{ with equality if } n > 0, \quad (12)$$

$$p(n - c) - \lambda \geq 0, \text{ with equality if } \mu > 0, \quad (13)$$

$$S \int V'(x' \lambda / p') (x' / p') dF(x', p' | m, p) - m \leq 0, \text{ with equality if } k > 0, \quad (14)$$

where  $\mu$  is a nonnegative multiplier.

Figure 1

We first solve (11)-(13) for  $c$ ,  $n$ , and  $p\mu$  as functions of  $\lambda/p$ . This is equivalent to finding the optimal consumption and labor supply for a fixed acquisition of money balances. The solution for  $p\mu$  will have the interpretation as the marginal cost (in units of foregone utility from consumption and leisure) of holding money. This solution is diagrammed in Fig. 1.

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<sup>6</sup> There is a question as to whether cash balances in this scheme are "trans- actions balances" or a "store of value." I think it is clear that the model under discussion is not rich enough to permit an interesting discussion of the distinctions between these, or-other, motives for holding money. On the other hand, all motives for holding money require that it be held for a -positive time interval before being spent: there is no reason to use money (as opposed to barter) if it is to be received for goods and then instantaneously exchanged for other goods. There is also the question of whether money "yields utility." Certainly the answer in this context is yes, in the sense that if one imposes on an individual the constraint that he cannot hold cash, his utility under an optimal policy is lower than it will be if this constraint is removed. It should be equally clear, however, that this argument does not imply that real or nominal balances should be included as an argument in the individual preference functions. The distinction is the familiar one between the utility function and the value of this function under a particular set of choices.

It is not difficult to show that, as Fig. I suggests, for any  $\lambda/p > 0$  (11)-(13) may be solved for unique values of  $c$ ,  $n$ , and  $P \mu$ . As  $\lambda/p$  varies, these solution values vary in a continuous and (almost everywhere) continuously differentiable manner. From the noninferiority assumptions (4), it follows that as  $\lambda/p$  increases,  $n$  increases and  $c$  decreases. The solution value for  $p^L$ , which we denote by  $h(\lambda/p)$  is, positive, increasing, and continuously differentiable. As  $\lambda/p$  tends to zero,  $h(\lambda/p)$  tends to a positive limit,  $h(0)$ .

Substituting the function  $h$  into (14), one obtains

$$h(\lambda/p) (1/p) S V' (x' \lambda/p') (x'/p') dF(x', p' | m, p), \quad (15)$$

with equality if  $\lambda > 0$ . After multiplying through by  $p$ , (15) equates the marginal cost of acquiring cash (in units of current utility foregone) to the marginal benefit (in units of expected future utility gained). Implicitly, (15) is a demand function for money, relating current nominal quantity demanded,  $X$ , to the current and expected future price levels.

#### 4 Expectations and a Definition of Equilibrium

Since the two markets in this economy are structurally identical, and since within a trading period there is no communication between them, the economy's general (current period) equilibrium may be determined by determining equilibrium in each market separately. We shall do so by equating nominal money demand (as determined in section 3) and nominal money supply in the market which receives a fraction  $\theta/2$  of the young. Equilibrium in the other market is then determined in the same way, with  $\theta$  replaced by  $2 - \theta$ , and aggregate values of output and prices are determined in the usual way by adding over markets. This will be carried out explicitly in section 6.

At the beginning of the last section, we observed that money will be supplied inelastically in each market. The total money supply, after transfer, is  $Nmx$ . Following the convention adopted in section 1,  $Nmx/2$  is supplied in each market. Thus in the market receiving a fraction  $\theta/2$  of the young, the quantity supplied per demander is  $(Nmx/2)/(\theta N/2) = mx/\theta$ . Equilibrium requires that  $\lambda = mx/\theta$ , where  $\lambda$  is quantity demanded per age-0 person. Since  $mx/\theta > 0$ , substitution into (15) gives the equilibrium condition

$$H(mx/\theta p) (1/p) = S V' (mxx'/\theta p') (x'/p') dF(x', p' | m, p). \quad (16)$$

Equation (16) relates the current period price level to the (unknown) future price level,  $p'$ . To "solve" for the market clearing price  $p$  (and hence to obtain the current equilibrium values of employment, output, and consumption)  $p$  and  $p'$  must be linked. This connection is provided in the definition of equilibrium stated below, which is motivated by the following considerations.

First, it was remarked earlier that in some (not very well defined) sense the state of the economy is fully described by the three variables  $(m, x, \theta)$ . That is, if at two different points in calendar time the economy arrives at a particular state  $(m, x, \theta)$  it is reasonable to expect it to behave the same way both times, regardless of the route by which the state was attained each time. If this is so, one can express the equilibrium price as a function  $p(m, x, \theta)$  on the space of possible states and similarly for the equilibrium values of employment, output, and consumption.

Second, if price can be expressed as a function of  $(m, x, \theta)$ , the true probability distribution of next period's price,  $p' = p(m', x', \theta) = p(mx, x',$

$\theta'$ ) is known, conditional on  $m$ , from the known distributions of  $x$ ,  $x'$ , and  $\theta'$ . Further information is also available to traders, however, since the current price,  $p(m, x, \theta)$ , yields information on  $x$ . Hence, on the basis of information available to him, an age-0 trader should take the expectation in (16) [or (15)] with respect to the joint distribution of  $(m, x, x', \theta')$  conditional on the values of  $m$  and  $p(m, x, \theta)$ , or treating  $m$  as a parameter, the joint distribution of  $(x, x', \theta')$  conditional on the value of  $p(m, x, \theta)$ . Denote this latter distribution by  $G(x, x', \theta' | p(m, x, \theta))$ .<sup>7</sup> -

We are thus led to the following

Definition. An equilibrium price is a continuous, nonnegative function  $p(\cdot)$  of  $(m, x, \theta)$ , with  $mx/\theta p(m, x, \theta)$  bounded and bounded away from zero, which satisfies:

$$h [mx/\theta p(m, x, \theta)] (1/p(m, x, \theta)) = S \int V' [mx'x'/\theta p(m, x', \theta')] (x'/p(m, x', \theta')) dG(x, x', \theta' | p(m, x, \theta)). \quad (17)$$

Equation (17) is, of course, simply (16) with  $p$  replaced by the value of the function  $p(\cdot)$  under the current state,  $(m, x, \theta)$ , and  $p'$  replaced by the value of the same function under next period's state  $(m, x', \theta')$ . In addition, we have dispensed with unspecified distribution  $F$ , taking the expectation instead with respect to the well-defined distribution  $G$ .<sup>8</sup>

In the next section, we show that (17) has a unique solution and develop the important characteristics of this solution. The more difficult mathematical issues will be relegated to the appendix.

## 5 Characteristics of the Equilibrium Price Function

We proceed by showing the existence of a solution to (17) of a particular form, then showing that there are no other solutions, and finally by characterizing the unique solution. As a useful preliminary step, we show:

Lemma 1. If  $p(\cdot)$  is any solution to (17), it is monotonic in  $x/\theta$  in the sense that for any fixed  $m$ ,  $x_0/\theta_0 > x_1/\theta_1$  implies  $p(m, x_0, \theta_0) \neq p(m, x_1, \theta_1)$ .

Proof. Suppose to the contrary that  $x_0/\theta_0 > x_1/\theta_1$  and  $p(m, x_0, \theta_0) = p(m, x_1, \theta_1) = p_0$  (say). Then from (17),

$$h (mx_0/\theta_0 p_0) (1/p_0) = S \int V' [mx_0 x'/\theta_0 p(m, x', \theta')] (x'/p(m, x', \theta' | p_0)) .$$

and

$$h (mx_1/\theta_1 p_0) (1/p_0) = S \int V' [mx_1 x'/\theta_1 p(m, x', \theta')] (x'/p(m, x', \theta' | p_0)) .$$

<sup>7</sup> The assumption that traders use the correct conditional distribution in forming expectations, together with the assumption that all exchanges take place at the market clearing price, implies that markets in the economy are efficient, as this term is defined by Roll [9]. It will also be true that price expectations are rational in the sense of Muth [7].

<sup>8</sup> The restriction, embodied in this definition, that price may be expressed as a function of the state of the economy appears innocuous but in fact is very strong. For example, in the models of Cass and Yaari without storage, the state of the economy never changes, so the only sequences satisfying the definition used here are constant sequences (or stationary schemes, in the terminology of [1]).

Since  $h$  is strictly increasing while  $VI$  is strictly decreasing, these equalities are- contradictory. This completes the proof.

In view of this Lemma, the distribution of  $(x, x', \theta')$  conditional on  $p(m, x, \theta)$  is the same as the distribution conditional on  $x/\theta$  for all solution functions  $p(\cdot)$ , a fact which vastly simplifies the study of (17).

It is a plausible conjecture that solutions to (17) assume the form  $p(m, x, \theta) = m \varphi(x/\theta)$ , where  $\varphi$  is a continuous, nonnegative function.<sup>9</sup> If this is true, the function  $\varphi$  satisfies [multiplying (17) through by  $mx/\theta$  and substituting]:

$$h [] \tag{18}$$

Let us make the change of variable  $z = x/\theta$ , and  $z' = x'/\theta'$ , and let  $H(z, \theta)$  be the joint density function of  $z$  and  $\theta$  and let  $H(z, 0)$  be the density of  $\theta$  conditional on  $z$ . Then (18) is equivalent to:

$$h[] \tag{19}$$

Equations (17) and (19) are studied in the appendix. The result of interest is:

Theorem 1. Equation(19)has exactly one continuous solution  $\varphi(z)$ on  $(0, \infty)$  with  $z/\varphi(z)$  bounded. The function  $\varphi(z)$  is strictly positive and continuously differentiable. Further,  $m\varphi(x/\theta)$  is the unique equilibrium price function.

Proof. See the appendix.

We turn next to the characteristics of the solution function  $\varphi$ . It is convenient to begin this study by first examining two polar cases, one in which  $\theta = 1$  with probability one, and a second in which  $x = 1$  with probability one.

The first of these two cases may be interpreted as applying to an economy in which all trading place in a single market, and no non- monetary disturbances are present. Then  $z$  is simply equal to  $x$  and, in view of Lemma 1, the current value of  $x$  is fully revealed to traders by the equilibrium price. It should not be surprising that the- following classical neutrality of money theorem holds.

Theorem 2. Suppose  $\theta = 1$  with probability one. Let  $y^*$  be the unique solution to

$$h(y) = V'(y). \tag{20}$$

Then  $p(m, x, \theta) = mx/y^*$  is the unique solution to (17).

Proof. We have observed that  $h$  is increasing and  $VI$  is decreasing, tending to 0 as  $y$  tends to infinity by (8). By (7),  $h(0) < V'(0)$ . Hence (20) does have a unique solution,  $y^*$ . It is clear that  $V(z) = zly^*$  satisfies (19). By Theorem 1, it is the only solution and  $mxly^*$  is the unique solution to (17).

The second polar case, where  $x$  is identically 1, may be interpreted as applying to an economy with real disturbances but with a perfectly stable monetary policy. In this case,  $z = 1/\theta$ , so that the current market price reveals  $\theta$  to all traders. It is convenient to let  $\Psi(\theta) = 1/[\theta\varphi(1/\theta)]$

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<sup>9</sup> To decide whether it is plausible that  $m$ . should factor out of the equilibrium price function, the reader should ask himself. what are the consequences of a fully announced change in the quantity of money which does not alter the distribution of money over persons? To see why only the ratio of  $x$  to  $\theta$  affects price, recall that  $x/\theta$  alone determines the demand for goods facing each individual producer.

so that (19) becomes:

$$h[\theta] \tag{21}$$

Denote the right side of (21) by  $m(\theta)$ . Then

$$M'(\theta) =$$

(suppressing the arguments of  $V'$  and  $VI$ ). The elasticity of  $m(\theta)$  is therefore

$$\theta m'(\theta) =$$

where

$$w(\theta, \theta') =$$

Clearly,  $w(\theta, \theta') \geq 0$  and  $\int w(\theta, \theta') d\theta' = 1$ . From (5) and (6)

$$0 < (VI) - V'' - f_p(0, \theta) + VI - I < 1.$$

Hence  $-\theta m'(\theta)/m(\theta)$  is a mean value of terms between 0 and 1, so that

$$-1 < \theta m'(\theta)/m(\theta) < 0. \tag{22}$$

Now differentiating both sides of (21), we have

$$[h'(\theta) + h]T'(\theta) = m'(\theta),$$

which using (22) and the fact that  $h$  is increasing implies

$$-1 < W(\theta) < 0. \tag{23}$$

Recalling the definition of  $Y(\theta)$  in terms of  $j(\theta)$ , it is readily seen that (23) implies

$$0 < \theta v - \theta < 1.$$

We summarize the discussion of this case in

**Theorem 3.** Suppose  $x = I$  with probability one. Then (17) has a unique solution  $p(m, x, \theta) = m j(1/\theta)$  where  $j$  is a continuously differentiable function, with an elasticity between zero and one.

If the factor disturbing the economy is exclusively monetary, then current price will adjust proportionally to changes in the money supply. Money is neutral in the short run, in the classical sense that the equilibrium level of real cash balances, employment, and consumption will remain unchanged in the face even of unanticipated monetary changes. These, in words, are the implications of Theorem 2. If, on the other hand, the forces disturbing the economy are exclusively real, the money supply being held fixed, disturbances will have real consequences. Those of the young generation who find themselves in a market with few of their cohorts (in a market with a low  $\theta$ , or a high  $z$ -value) obtain what is in effect a lower price of future consumption. Theorem 3, resting on the assumptions of income and substitution effects set out in section 3, indicates that they will distribute all of this gain to the future, holding



higher real balances. This attempt is partially frustrated by a rise in the current price level.

Returning to the general case, in which both  $x$  and  $\theta$  fluctuate, it is clear that the current price informs agents only of the ratio  $x/\theta$  of these two variables. Agents cannot discriminate with certainty between real and monetary changes in demand for the good they offer, but must instead make inferences on the basis of the known distributions  $f(x)$  and  $g(\theta)$  and the value of  $x/\theta$  revealed by the current price level. It seems reasonable that their behavior will somehow mix the strategies described in Theorems 2 and 3, since a high  $x/\theta$  value indicates a high  $x$  and a low  $\theta$ .

Unfortunately this last statement, aside from being imprecise, is not true, as one can easily show by example.<sup>10</sup> Hence we wish to impose additional restrictions on the densities  $f$  and  $g$ , with the aim of assuring that, first, for any fixed  $\theta$ ,  $\Pr(\theta < \theta | x/\theta = z)$  is an increasing function of  $z$ , and, second, that for any fixed  $x$ ,  $\Pr(x < x | x/\theta = z)$  is a decreasing function of  $z$ . Using  $H(z, \theta)$  as above to denote the density of  $\theta$  conditional on  $x/\theta = z$  the first of these probabilities is

$$F(z, \theta) = \int_0^\theta H(z, \theta) d\theta,$$

while the second, in terms of the same function  $F$ , is  $F(z, x/z)$ . The desired restriction is then found (by differentiating with respect to  $z$ ) to be:

$$0 < F_z(z, 0) < z \tag{24}$$

for all  $(z, \theta)$ . We proceed, under (24), with a discussion analogous to that which precedes Theorem 3.

Let

$$m(\theta)$$

where, as in the proof of Theorem 3,  $m(\theta)$  is positive with an elasticity between -1 and 0.

Then (19) may be written

$$h \tag{25}$$

Denote the right side of (25) by  $G(z)$ . Then integrating by parts,

$$G(z) = m(2) - \int_0^z f m'(\theta) F(z, \theta) d\theta$$

where it will be recalled that 2 is the upper-limit of the range of  $\theta$ .

Then

$$G'(z) = \int_0^z f m'(\theta) F_z(z, \theta) d\theta > 0,$$

by the first inequality of (24). Continuing,

$$z G'(z) / G(z) =$$

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<sup>10</sup> For example, let  $x$  take only the values 1 and 1.05 and let  $\theta$  be either 0.5 or 1.5. Then a decrease of  $x/\theta$  from 2.0 to 0.7 implies (with certainty) an increase in  $x$  from 1 to 1.05. It is not difficult to construct continuous densities  $f$  and  $g$  which exhibit this sort of behavior.

where  $w(z, 0) = \int_0^1 [f_m(0), I_\theta(z, 0) d0] - I_m(0), q(z, 0)$ . Hence, applying (24) again,  
 $0 > z_j(z)/j(z) < 1$  (26)

We summarize the discussion of this case in

**Theorem 4.** Suppose the function  $F(z, \theta)$ , obtained from the densities  $i(x)$  and  $g(0)$ , satisfies the restriction (24). Then (17) has a unique solution  $p(m, x, \theta) = mV(x/\theta)$ , where  $V$  is a continuously differentiable function, with an elasticity between zero and one.

Theorems 2-4 indicate that, within this framework, monetary changes have real consequences only because agents cannot discriminate perfectly between real and monetary demand shifts. Since their ability to discriminate should not be altered by a proportional change in the scale of monetary policy, intuition suggests that such scale changes should have no real consequences. We formalize this as a corollary to Theorem 4:

**Corollary.** Let the hypotheses of Theorem 4 hold, but let the transfer variable be  $y = kx$ , where  $k$  is a positive constant. Then the equilibrium price is  $p(m, y, 0) = \frac{1}{k} p(y/k, x, 0) = mp(x/\theta)$ , where  $p$  is as in Theorem 4.

**Proof.** In the derivation of (19), let  $z = y/\theta = x/\theta$ .

## 6 Positive Implications of the Theory

In the previous section we have studied the determination of price in one of the markets in this two market economy: the one which received a fraction  $\theta/2$  of producers. Excluding the limiting case in which the disturbance is purely monetary, this price function was found to take the form  $m(p(x/\theta))$ , where  $p(x/\theta)$  is positive with an elasticity between zero and one. Recalling the study of the individual producer-consumer in section 3, this price function implies an equilibrium employment function  $n(x/\theta)$ , where  $n'(x/\theta) > 0$ .<sup>11</sup> That is, increases in demand induce increases in real output. Since the two markets are identical in structure, equilibrium price in the other market will be  $mp(x/(2 - \theta))$  and employment will be  $n(x/(2 - \theta))$ . In short, we have characterized behavior in all markets in the economy under a possible states.

With this accomplished, it is in order to ask whether this behavior does in fact resemble certain aspects of the observed business cycle. One way of phrasing this question is: how would citizens of this economy describe the ups and downs they experience?<sup>12</sup>

Certainly casual observers would describe periods of higher than average  $x$ -values (monetary expansions) as "good times" even, or perhaps especially, in retrospect. The older generation will do so with good reason: they receive the transfer, and it raises their real consumption levels to higher than average levels. The younger generation will similarly approve a monetary expansion as it occurs: they perceive it only through a higher-than-average price of the goods they are selling in their real wealth. In which, on average, means an increase future, they will, of course, be disappointed (on average) in the real

<sup>11</sup> The analysis of section 3 showed that if age-0 consumers wish to accumulate more real balances, they will finance this accumulation in part by supplying more labor. In section 5 it was shown that equilibrium per capita real balances,  $[\theta \square(x/\theta)]x$ , rise with  $x/\theta$ . These two facts together imply  $n'(x/\theta) > 0$ .

<sup>12</sup> The following discussion, while I hope it is suggestive, is not intended to be a substitute for econometric evidence.

consumption their accumulated balances provide. Yet there is no reason for them to attribute this disappointment to the previous expansion; it would be much more natural to criticize the current inflation. This criticism could be expected to be particularly severe during periods, which will regularly arise, when inflation continues at a higher than average rate while real output declines.<sup>13</sup> To summarize, in spite of the symmetry between ups and downs built into this simple model, all participants will agree in viewing periods of high real output as better than other periods.<sup>14</sup>

Less casual observers will similarly be misled. To see why, we consider the results of fitting a variant of an econometric Phillips curve on realizations generated by the economy described above. Let  $Y_t$  denote real GNP (or employment) in period  $t$ , and let  $P_t$  be the implicit GNP deflator for  $t$ . Consider the regression hypothesis

$$\ln Y_t = b_0 + b_1(\ln P_t - \ln P_{t-1}) + E_t, \quad (27)$$

where  $E_1, E_2, \dots$  is a sequence of independent, identically distributed, random variables with 0 mean. Certainly a positive estimate for  $b_1$  would, provided the estimated residuals do not violate the hypothesis, be interpreted as evidence for the existence of a "trade-off" between inflation and real output. By this point, it should be clear intuitively that there is no such trade-off in the model under study, yet  $b_1$  will turn out to be positive. We next develop the latter point more explicitly.

We have:

$$Y_t = N_t n(\mu) \phi(x_t) \quad (28)$$

and,

$$P_t = N_t \phi(x_t) \quad (29)$$

Let  $\mu = E[\ln(x)] = \int \ln(x) f(x) dx$ . Regarding the logs of the right sides of (28) and (29) as functions of  $\ln(x_t)$  and  $\ln(P_{t-1})$ , expanding these about  $(\mu, 0, 1)$  and discarding terms of the second order and higher we obtain the approximations:

$$\ln(Y_t) = \ln(N_t) + \ln[n(\mu)] + \eta_n [\ln x_t - \mu] \quad (30)$$

and

$$\ln(P_t) - \ln(P_{t-1}) = \eta_\phi \ln x_t + (1 - \eta_\phi) \ln x_{t-1} \quad (31)$$

where  $\eta_n$  and  $\eta_\phi$  are the elasticities of the functions  $n$  and  $\phi$ , respectively, evaluated at  $\mu$ .

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<sup>13</sup> The term "regularly arise" is appropriate. The current real output level, relative to "normal," depends only on the current monetary expansion. The current inflation rate, however, depends on the current and previous period's monetary expansion. Thus a large expansion followed by a modest contraction will occur (though perhaps infrequently) and will result in the situation described in the text.

<sup>14</sup> This unanimity rests, of course, on the assumption that new money is introduced so as never to subject cash holders to a real capital loss. If transfers were, say, randomly distributed over young and old, there would be a group among the old which perceives monetary expansion as harmful.

Using (30) and (31), one can compute the approximate<sup>15</sup> probability limit of the estimated coefficient  $\hat{\alpha}$ , of (27). It is the covariance of  $\ln(Y_t)$  and  $\ln(\text{PilPt}_t)$ , divided by the variance of the latter, or  $\text{cov}(\ln Y_t, \ln \text{PilPt}_t) / \text{var}(\ln \text{PilPt}_t) > 0$ .

The estimated residuals from this regression will exhibit negative serial correlation. By adding  $\ln(Y_t)$  as an additional variable, however, this problem is eliminated and a near perfect fit is obtained [cf. (30) and (31)]. The coefficient on the inflation rate remains positive.<sup>16</sup>

To summarize this section, we have deliberately constructed an economy in which there is no usable trade-off between inflation and real output. Yet the econometric evidence for the existence of such trade-offs is much more convincing here than is the comparable evidence from the real world.

## 7 Policy Considerations

Within the framework developed and studied in the preceding sections, the choice of a monetary policy is equivalent to the choice of a density function  $f$  governing the stochastic rate of monetary expansion. Densities  $f$  which are concentrated on a single point correspond to fixing the rate of monetary growth at a constant percentage rate  $k$ . Following Friedman, we shall call such a policy a  $k$ -percent rule. Any other policy implies random fluctuations about a constant mean. Since (as far as I know) no critic of a  $k$ -percent rule consciously advocates a randomized policy in its stead, there is little interest pursuing a study of monetary policies within the restricted class available to us in this context. We can, however, show that if a  $k$ -percent rule is followed the competitive allocation will be Pareto-optimal. This demonstration will occupy the remainder of this section.

For the case of a constant money supply ( $x = -1$ ) there is an equilibrium price function  $m(p)$ , the properties of which are given in Theorem 3. Corresponding to this price function are functions  $c(\theta)$ ,  $n(\theta)$  which give the equilibrium values of consumption and labor supply of the young for each possible state of the world,  $\theta$ . Since product is exhausted, these imply an average per capita consumption level for the old in the same market: 17

$$c(\theta) =$$

By the Corollary to Theorem 4, this allocation rule  $\{Z^F(\theta), i(\theta), F'(\theta)\}$  will be followed if monetary policy follows any  $k$ -percent rule. We wish to compare the efficiency of this rule to alternative (nonmarket) allocation rules  $\{f^e(\theta), n(\theta), c'(\theta)\}$ .

The individuals whose tastes are to be taken into account are the successive generations inhabiting the model economy. If we continue to ignore calendar time (to treat present and future generations symmetrically) each generation can be indexed by the states of nature  $(\theta, \theta')$  which prevail during its lifetime. This leads to the notion that one allocation is superior to another in a Pareto sense if it is preferred uniformly over all possible states, or to the following

**Definition.** An allocation rule  $\{c(\theta), n(\theta), c'(\theta)\}$  is Pareto-optimal if it satisfies

<sup>15</sup> Because (30) and (31) are approximations.

<sup>16</sup> It is interesting to note that if one formulates a distributed lag version of the Phillips curve, as Rapping and I have done in [6], one will obtain a positive estimated long-run real output-inflation trade-off even if a model of the above sort is valid.

$$c(\theta) + \int c'(\theta) - n(\theta), \quad c(\theta), n(\theta), c'(\theta) \geq 0 \quad (32)$$

(is feasible) for all  $0 < \theta < 2$ , and if there is no feasible allocation rule  $\{c(\theta), n(\theta), c'(\theta)\}$  such that

$$U[c(\theta), n(\theta)] \quad (33)$$

$$c'(\theta) \leq F'(\theta), \quad (34)$$

for all  $\theta$ , with strict inequality in either (33) or (34) over some subset of  $(0, 2)$  assigned positive probability by  $g(\theta)$ .

We then have:

Theorem 5. The equilibrium  $\{c(\theta), n(\theta), c'(\theta)\}$ , which arises under a  $k$ -percent rule, is Pareto-optimal.

Proof. Suppose, to the contrary, that an allocation  $\{c(\theta), n(\theta), c'(\theta)\}$  satisfying (32)-(34) exists. Recall from sections 3 and 5 that the

$$\max_{c, n} \int U(c, n) + \int V \quad g(\theta) d\theta$$

subject to  
 $[n - c] - X \geq 0$

is uniquely solved by  $c(\theta)$ ,  $n(\theta)$  and  $X = m_0$ . Hence  $ZF'(\theta)$

Now using (32), if

$$\begin{aligned} X(\theta) &= n(\theta) - c(\theta) \\ c'(\theta) &= m_0 - c(\theta) \end{aligned}$$

then  $c(\theta), n(\theta), X(\theta)$  is feasible for this problem. Since (if it differs from the equilibrium) it cannot be optimal for this problem,

$$\int U[c(\theta), n(\theta)] + \int V \quad g(\theta) d\theta$$

$$> \int U[c(\theta), n(\theta)] + \int V \quad g(\theta) d\theta$$

By (33), this implies

$$S \quad (35)$$

But by (34),  $c'(\theta) \leq F'(\theta)$ , so that

$$\int [c(\theta), n(\theta), X(\theta)] \quad g(\theta) d\theta - \int [c(\theta), n(\theta), X(\theta)] \quad g(\theta) d\theta$$

This contradicts (35), contradicting the assumed superiority of  $\{c(\theta), n(\theta), c'(\theta)\}$ , and completes the proof.

Two features of this discussion should perhaps be reemphasized. First, Theorem 5 does not compare resource allocation under a  $k$ -percent rule to allocations which result from other monetary policies. In general, the latter allocations will be randomized, in the sense that allocation for given  $\theta$  will be stochastic. It does compare allocation under a  $k$ -percent rule to other

nonrandomized (and thus nonmarket) allocation rules. Second, our discussion of optimality takes the market information structure of the economy as a physical datum. Obviously, if the two markets can costlessly be merged, superior resource allocation can be obtained.

## 8 Conclusion

This paper has been an attempt to resolve the paradox posed by Gurley [41, in his mild but accurate parody of Friedmanian monetary theory: "Money is a veil, but when the veil flutters, real output sputters." The resolution has been effected by postulating economic agents free of money illusion, so that the Ricardian hypothetical experiment of a fully announced, proportional monetary expansion will have no real consequences (that is, so that money is a veil). These rational agents are then placed in a setting in which the information conveyed to traders by market prices is inadequate to permit them to distinguish real from monetary disturbances. In this setting, monetary fluctuations lead to real output movements in the same direction.

In order for this resolution to carry any conviction, it has been necessary to adopt a framework simple enough to permit a precise specification of the information available to each trader at each point in time, and to facilitate verification of the rationality of each trader's behavior. To obtain this simplicity, most of the interesting features of the observed business cycle have been abstracted from, with one notable exception: the Phillips curve emerges not as an unexplained empirical fact, but as a central feature of the solution to a general equilibrium system.

## Appendix Proof of Theorem I

We first show the existence of a unique solution to (19). Define  $\phi(z)$  by

$$\phi(z) = h(z) - z$$

Let  $G$  be the inverse of the function  $h(x)$ , so that  $z = G(\phi(z))$ . The function  $G(x)$  is positive for all  $x > 0$ , and satisfies

$$\lim_{x \rightarrow 0} G(x) = 0, \tag{A1}$$

and

$$0 < x \leq G(x) < \frac{x}{1-a} \tag{A2}$$

Let  $G_2(x) = V'(x)x$ .  $G_2(x) > 0$  for all  $x > 0$  and, repeating (5) and (6),  $0 < 1 - a < 1$ .  $G_2(x)$  (A3)

In terms of the functions  $\phi$ ,  $G$ , and  $G_2$  (19) becomes

$$\phi(z) = \int_0^z G_2[G(\phi(x'))] \phi'(z', 0) H(z', 0) dz'.$$

(A4) Let  $S$  denote the space of bounded, continuous functions on

$[-1, 1]$ , normed by

$$\|f\| = \sup_{z \in [-1, 1]} |f(z)|.$$

Define the operator T on S by

$Tf = \int_0^1 G(e, z) \cdot [R(z, 0), H(z, 0)] dz$ . In terms of T, (A4) is

$InT = Tin41$ .

(A5) We have:

Lemma2. Tisacontractionmapping:foranyfgES,  $JITF - Tgll = (I - a)llf - gll$ .  
Proof.

In  $w(0, z, 0, z) = \int_0^1 [G(e - qz)] dz$

$1ITf - Tgll = \sup_{I, f} \int_0^1 [G(e - qz)] dz$

where

$w(0, z, 0, z)$

$= \int_0^1 [G(e - qz)] dz - \int_0^1 [G(e - qz)] dz$ .

Since  $w(0, z, 0, z) > 0$  and  $f w dz$

I we have, continuing,

$\int_0^1 G(e - qz) dz = \int_0^1 G(e - qz) dz$  (A6)  $1ITf - Tgll = \sup_{I, f} \int_0^1 [G(e - qz)] dz$

Now

$a \int_0^1 G(e) dz = \int_0^1 G(e) dz$

By (A3), the first of these factors is between 0 and  $I - a$ . By (A2), the second factor is between 0 and 1. Since these observations are valid for all  $(x, 0, 6)$ , application of the mean value theorem to the right side of (A6) gives

$JITF - Tgll = (I - a)llf - gll$ , which completes the proof.

It follows from Lemma 2 and the Banach fixed point theorem that

the equation  $Tf = f$  has a unique bounded, continuous solution  $f$ . Then  $T(z) = ef(z)$  is the unique solution to (A4). Clearly  $4r(z)$  is positive, bounded, and bounded away from zero. It follows that  $G, [4r(z)]$  has these properties, and hence that  $p(z) = z / (G, [T(z)])$  is the function referred to in Theorem 1. Clearly  $m(p(x))$  is an equilibrium price function [satisfies (17)]. In view of Lemma 1, any solution  $p(m, x, 0)$  must satisfy:

$tnx h = nx$   
 $op(m, x, 0) = op(m, x, 0)$   
 $0 < x < 1$   $nzfx? = dG f V, \quad 0 \leq I$

Now let  $T(m, x, 0) = h[mx1(Op(m, x, 0))] mx1[Op(m, x, 0)]$ . Proceeding as before, one finds that there is only one bounded solution  $IP(m, x, 0)$ . This proves Theorem 1.

### Notes

1. The usefulness of this model as a framework for considering problems in monetary theory is indicated by the work of Cass and Yaari [1, 2].

2. This is not quite right. If members of the younger generation were risk preferrers, they could and would exchange claims on future consumption among themselves so as to increase variance. This possibility will be filed out in the next section.

3. This device of viewing traders as randomly allocated over distinct markets serves two purposes. First, it provides a setting in which information is imperfect in a specific (and hence analyzable) way. Second, random in the allocation of traders provides a source of relative price variation. This could as well have been achieved by postulating random taste or technology shifts, with little effect on the structure of the model.

4. This somewhat artificial assumption, like the absence of capital goods and the serial independence of shocks, is part of an effort to keep the laws governing the transition of the economy from state to state as simple as possible. In general, I have tried to abstract from all sources of persistence of fluctuations, in order to focus on the nature of the initial disturbances.

5. The restrictions (4) and (5) are similar to those utilized in an econometric study of the labor market conducted by Rapping and myself, [5]. Their function here is the same as it was in [5]: to assure that the Phillips curve slopes the "right way."

6. There is a question as to whether cash balances in this scheme are "transactions balances" or a "store of value." I think it is clear that the model under discussion is not rich enough to permit an interesting discussion of the distinctions between these, or-other, motives for holding money. On the other hand, all motives for holding money require that it be held for a -positive time interval before being spent: there is no reason to use money (as opposed to barter) if it is to be received for goods and then instantaneously exchanged for other goods. There is also the question of whether money "yields utility." Certainly the answer in this context is yes, in the sense that if one imposes on an individual the constraint that he cannot hold cash, his utility under an optimal policy is lower than it will be if this constraint is removed. It should be equally clear, however, that this argument does not imply that real or nominal balances should be included as an argument in the individual preference functions. The distinction is the familiar one between the utility function and the value of this function under a particular set of choices.

7. The assumption that traders use the correct conditional distribution in forming expectations, together with the assumption that all exchanges take place at the market clearing price, implies that markets in t s economy are efficient, as this term is defined by Roll [9]. It will also be true that price expectations are rational in the sense of Muth [7].

8. The restriction, embodied in this definition, that price may be expressed as a function of the state of the economy appears innocuous but in fact is very strong. For example, in the models of Cass and Yaari without storage, the state of the economy never changes, so the only sequences satisfying the definition used here are constant sequences (or stationary schemes, in the terminology of [I]).

9. To decide whether it is plausible that  $m$ . should factor out of the equilibrium price function, the reader should ask himself. what are the consequences of a fully announced change in the quantity of money which does not alter the distribution of money over persons? To see why only the ratio of  $x$  to



0 affects price, recall that  $x/\theta$  alone determines the demand for goods facing each individual producer.

10. For example, let  $x$  take only the values 1 and 1.05 and let  $\theta$  be either 0.5 or 1.5. Then a decrease of  $x/\theta$  from 2.0 to 0.7 implies (with certainty) an increase in  $x$  from 1 to 1.05. It is not difficult to construct continuous densities  $f$  and  $g$  which exhibit this sort of behavior.

11. The analysis of section 3 showed that if age-0 consumers wish to accumulate more real balances, they will finance this accumulation in part by supplying more labor. In section 5 it was shown that equilibrium per capita real balances,  $[\theta \phi(x/\theta)]^{-1}x$ , rise with  $x/\theta$ . These two facts together imply  $n'(x/\theta) > 0$ .

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13. The term "regularly arise" is appropriate. The current real output level, relative to "normal," depends only on the current monetary expansion. The current inflation rate, however, depends on the current and previous period's monetary expansion. Thus a large expansion followed by a modest contraction will occur (though perhaps infrequently) and will result in the situation described in the text.

14. This unanimity rests, of course, on the assumption that new money is introduced so as never to subject cash holders to a real capital loss. If transfers were, say, randomly distributed over young and old, there would be a group among the old which perceives monetary expansion as harmful.

15. Because (30) and (31) are approximations.

16. It is interesting to note that if one formulates a distributed lag version of the Phillips curve, as Rapping and I have done in [6], one will obtain a positive estimated long-run real output-inflation trade-off even if a model of the above sort is valid.

17. The unequal distribution of money acquired during the first year of life (due to varying  $\theta$  values) creates two classes among the old. In general, then, no one will actually obtain the average consumption  $c'(\theta)$ . But a reallocation which receives the unanimous consent of the old in the market receiving a fraction  $\theta$  of producers is possible if and only if average consumption is increased. For our purposes, then, we can ignore the distribution of actual consumption about this average.

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